Convex Relaxations in Mixed-Integer Optimization

Methods and Control Applications

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Outline

1 Introduction

2 Optimization of Large Scale Systems

3 Example: Electric Vehicles Charging Coordination
What are Mixed–Integer Optimization Problems / Why do we look into them

- many practical and industrial systems entail **continuous quantities**
  - physical measurements of voltages
  - concentrations
  - and positions in space
  - as well as **discrete components**
    - on/off decisions
    - switches
    - and logic reasoning (if, or, ...)

- when the associated control/operation tasks are tackled with optimization, **Mixed-Integer Optimization Problems (MIPs)** arise

- but the additional flexibility has a price ...
Computational Issues – a Practical Perspective

- experience with instances from supply chain problem
- modest size, yet memory blew up

<table>
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<th># cust.</th>
<th># prod.</th>
<th>CPLEX time (sec)</th>
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</thead>
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<td>100</td>
<td>25</td>
<td>Min: 10.3</td>
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<tr>
<td></td>
<td></td>
<td>Avg: 63.5</td>
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<tr>
<td></td>
<td></td>
<td>Max: 202.5</td>
</tr>
<tr>
<td>300</td>
<td>75</td>
<td>out of memory</td>
</tr>
</tbody>
</table>

- computational requirements depend not only on structure and size of the problem, but also the data
  - bad in a control context

The thesis focuses on particular model structures that are of **practical interest**. For these, we derived **computationally attractive approximation schemes**, equipped with **guarantees**.
Thesis Content

Part I – Optimization of Large Scale Systems.
- Lagrangian Duality
- Applications:
  - Power Systems
  - Supply Chain Optimization

Part II – Robust Optimization of Uncertain Systems.
- Linear Programming Relaxations
- Applications:
  - Scheduling under Uncertainty
  - PWM Systems

References:
- [EEM-13]
- [MED-14]
- [CDC-14]
- [Submitted, Math. Prog.-13]
- [ACC-12]
- [ECC-13]
- [CDC-13]
# Thesis Content

## Part I – Optimization of Large Scale Systems.

- **Lagrangian Duality**
- **Applications:**
  - Power Systems
  - Supply Chain Optimization

## Part II – Robust Optimization of Uncertain Systems.

- **Linear Programming Relaxations**
- **Applications:**
  - Scheduling under Uncertainty
  - PWM Systems

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- [MED-14]
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- [ECC-13]
- [CDC-13]
Outline

1. Introduction

2. Optimization of Large Scale Systems

3. Example: Electric Vehicles Charging Coordination
we consider the problem

\[
P : \begin{cases} 
\min & \sum_i c_i x_i \\
\text{s.t.} & \sum_i H_i x_i \leq b \\
x_i \in X_i & i = 1, \ldots, l 
\end{cases}
\]

where

\[X_i = \{x_i \in \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i} | A_i x_i \leq d_i\}\]

large collection of subsystems

- subsystem model is \(X_i\) (mixed–integer)
- coupled by shared resources \(\rightarrow\) coupling constraints \(\sum_i H_i x_i \leq b\)

\# of subsystems \(l \gg\) \# of coupling constraints \(m\)
Problem’s Decomposition

Obtain decomposition using duality:

\[
\begin{align*}
\min_x & \quad \sum_{i \in I} c_i x_i \\
\text{s.t.} & \quad \sum_{i \in I} H_i x_i \leq b \\
& \quad x_i \in X_i
\end{align*}
\]

\[
\Rightarrow \quad \min_x \quad \sum_{i \in I} c_i x_i + \lambda' \left( \sum_{i \in I} H_i x_i - b \right)
\]

\[
\text{s.t.} \quad x_i \in X_i
\]

\[
\Rightarrow \quad \sum_{i \in I} \min_{x_i \in X_i} \left\{ c_i x_i + \lambda' (H_i x_i) \right\} - \lambda' b
\]

\[
\therefore d(\lambda)
\]

• Lagrangian dual (or outer) problem:

\[
\mathcal{D} : \begin{cases} 
\max & d(\lambda) \\
\text{s.t.} & \lambda \geq 0
\end{cases}
\]
Solutions to the inner problem

\[ d(\lambda) = \sum_{i \in I} \min_{x_i \in X_i} \left\{ c_i x_i + \lambda'(H_i x_i) \right\} - \lambda' b \]

Consider

\[ x_i(\lambda^*) \in \arg \min_{x_i \in X_i} \left\{ c_i x_i + \lambda^*(H_i x_i) \right\} \]

as candidate solution to \( P \)

Properties of \( x_i(\lambda^*) \):

• satisfy \( X_i \) constraints
• obtained “for free” as by-product of methods that solve \( D \)
• distributed computations
• generally \textbf{infeasible} in the MIP case!
  ▶ violate coupling constraints
Primal Recovery Scheme

- we show that in $x(\lambda^*)$ only $m$ subsystems may be “problematic”
  - technique based on Shapley–Folkman–Starr theorem [CDC '14]
  - or simplex tableaux argument [MED '13]
  - these arguments are used to show bounded duality gap [Ekeland '76, Bertsekas '83]

- so we propose to consider instead

$$\begin{align*}
\overline{P} : \begin{cases} \\
\min_x & \sum_{i \in I} c_i x_i \\
\text{s.t.} & \sum_{i \in I} H_i x_i \leq b - \rho \\
& x_i \in X_i \\
& \forall i \in I,
\end{cases}
\end{align*}$$

where

$$\rho = m \cdot \max_{i \in I} \left( \max_{x_i \in X_i} H_i x_i - \min_{x_i \in X_i} H_i x_i \right)$$

Theorem

Then $x(\overline{\lambda}^*)$ is feasible for $\overline{P}$. [under some uniqueness assumptions]
Performance of the Recovered Solutions

- under some technical assumption, ...

**Theorem**

The recovered solution $\tilde{x}(\tilde{\lambda}^*)$ is feasible and satisfies

$$J_P(x(\tilde{\lambda}^*)) - J^*_P \leq (m + \|\rho\|_\infty / \zeta) \cdot \left( \max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)$$

- if $J^*_P$ grows linearly with $|I|$, and $X_i$ uniformly bounded

$$\frac{J(x(\tilde{\lambda}^*)) - J^*_P}{J^*_P} \to 0 \quad \text{as} \quad |I| \to \infty$$
ρ scales with $m$ but not with $l$ – want to keep it as small as possible

• when couplings are determined by certain network topologies

• can safely use $\text{rank}([H_i]_{i \in I_k})$ instead of $m$
• generally possible to use $\text{rank}(H)$ instead of $m$
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1. Introduction

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3. Example: Electric Vehicles Charging Coordination
Electric Vehicle (EV) Charging Coordination [CDC ’14]

- expected increase of EV presence
- substantial additional stress on network & equipment
  ⇒ need charging coordination
- network administrator (DSO) can’t tackle each single unit
  ⇒ **EV aggregator**
Aggregator’s Role

- aggregator’s **control task** is to assign to each EV the time slots when charging can occur compatibly with...
- **local requirements**
  - required final State of Charge
  - fixed charge rates
  - battery capacity limits
- **global objectives**
  - network congestion avoidance (limits set by DSO)
  - “valley fill”, cost min., ...
Computational Experiments

- cast as large optimization problem
- solve using proposed method: duality + contraction
  - support extensions (e.g., vehicle–to–grid “V2G”)
- population up to 10’000 EVs
- computation times $\leq$ 10 sec (charge only)
  - greedy subproblem structure
Solutions – Charge and V2G

(a) reference tracking

(b) resulting “valley fill”

(c) network limits

(d) local requirements
Other Examples or Applications

• supply chains optimization – partial shipments [MED '14]
• power systems operation
  ▶ control of TCLs
  ▶ large fleet of generators
• portfolio optimization for small investors
• …
Questions?

- Prof. Manfred Morari
- Dr. Paul Goulart
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- Gregory Ledva
- Isik Ilber Sirmatel
- Marius Schmitt
- Prof. Debasish Chatterjee
- Prof. Federico Ramponi
- Dr. Peter Hokayem
- Dr. Apostolos Fertis
- Dr. Utz-Uwe Haus
- Dr. Alexander Fuchs
- Dr. Martin Herceg
- Robert Nguyen
References


Convex Relaxations of MIPs

- our methods are based on convex relaxations
- given pure binary problem

\[
P : \begin{cases} 
\min_x c^T x \\
\text{s.t.} \begin{align*} 
a_1 x & \leq b_1 \\
a_2 x & \leq b_2 \\
x & \in \{0, 1\}^n \end{align*}
\end{cases}
\]

- if you had to solve it, which relaxation would you pick?

\[
P_1 : \begin{cases} 
\min_x c^T x \\
\text{s.t.} \begin{align*} 
a_1 x & \leq b_1 \\
a_2 x & \leq b_2 \\
\text{conv}(x \in \{0, 1\}^n) \end{align*}
\end{cases}
\]

or

\[
P_2 : \begin{cases} 
\min_x c^T x \\
\text{s.t.} \begin{align*} 
a_1 x & \leq b_1 \\
\text{conv} & \begin{align*} 
a_2 x & \leq b_2 \\
x & \in \{0, 1\}^n \end{align*}
\end{align*}
\end{cases}
\]

or

\[
P_3 : \begin{cases} 
\min_x c^T x \\
\text{s.t.} \begin{align*} 
\text{conv} & \begin{align*} 
a_1 x & \leq b_1 \\
a_2 x & \leq b_2 \\
x & \in \{0, 1\}^n \end{align*}
\end{align*}
\end{cases}
\]
Strength of the Relaxations

\[
\begin{align*}
&\text{min. } \sum_{i \in I} c_i x_i \\
&\text{s.t. } \sum_{i \in I} H_i x_i \leq b \\
& \quad A_i x_i \leq d_i \\
& \quad x_i \in \{0, 1\}^{n_i}
\end{align*}
\]
Issues affecting inner solutions

\[
\begin{align*}
\text{min } & -x_1 \\
\text{s.t. } & x_1 - x_2 \leq 0.5 \\
& x_1 + x_2 \leq 1.5 \\
& x_1, x_2 \in \{0, 1\}
\end{align*}
\]
Issues affecting inner solutions

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\min & \quad -x_1 \\
\text{s.t.} & \quad x_1 - x_2 \leq 0.5 \\
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\[ \Rightarrow \max_{\lambda \geq 0} \min_{x_1, x_2 \in \{0, 1\}} -x_1 + \lambda_1 (x_1 - x_2 - 0.5) + \lambda_2 (x_1 + x_2 - 1.5) \]
Issues affecting inner solutions

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\[ \Rightarrow d^* = -1 \quad \lambda^* = (0.25, 0.25) \]
Issues affecting inner solutions

\[ \Rightarrow d^\star = -1 \quad \lambda^\star = (0.25, 0.25) \]
Issues affecting inner solutions

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\min \quad -x_1 \\
\text{s.t.} \quad x_1 - x_2 \leq 0.5 \\
x_1 + x_2 \leq 1.5 \\
x_1, x_2 \in \{0, 1\}
\]

\[
\Rightarrow \quad \max_{\lambda \geq 0} \min_{x_1, x_2 \in \{0, 1\}} \quad -x_1 + \lambda_1 (x_1 - x_2 - 0.5) + \lambda_2 (x_1 + x_2 - 1.5)
\]

\[
\Rightarrow \quad d^* = -1 \quad \lambda^* = (0.25, 0.25)
\]}
Illustration of the Problems with the L-Relaxation

\[
\begin{align*}
\text{min}_x & \quad 3x_1 - x_2 \\
\text{s.t.} & \quad x_1 - x_2 \geq -1 \\
& \quad -x_1 + 2x_2 \leq 5 \\
& \quad 3x_1 + x_2 \geq 3 \\
& \quad 6x_1 + x_2 \leq 15 \\
& \quad x_1, x_2 \geq 0 \\
& \quad x_1, x_2 \in \mathbb{Z}
\end{align*}
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\[d(\lambda) = \min_{x \in X} (3x_1 - x_2 + \lambda(-1 - x_1 + x_2))\]
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& \quad 3x_1 + x_2 \geq 3 \\
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& \quad x_1, x_2 \geq 0 \\
& \quad x_1, x_2 \in \mathbb{Z}
\end{align*}
\]

\[
d(\lambda) = \min_{x \in \mathbb{N}} (3x_1 - x_2 + \lambda(-1 - x_1 + x_2))
\]

\[
\Rightarrow \quad \max_{\lambda \geq 0} d(\lambda)
\]
Illustration of the Problems with the L-Relaxation

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\begin{align*}
\min_x & \quad 3x_1 - x_2 \\
\text{s.t.} & \quad x_1 - x_2 \geq -1 \\
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\end{align*}
\]

\[d(\lambda) = \min_{x \in \mathcal{X}} (3x_1 - x_2 + \lambda(-1 - x_1 + x_2))\]

\[\Rightarrow \max_{\lambda \geq 0} d(\lambda) \Rightarrow \lambda^* = \frac{5}{3}\]
Illustration of the Problems with the L-Relaxation

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\[\Rightarrow \quad \max_{\lambda \geq 0} d(\lambda) \Rightarrow \lambda^* = 5/3\]

\[\Rightarrow \quad \mathcal{X}(\lambda^*) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}\]
Illustration of the Problems with the L-Relaxation

\[ \min_x \quad 3x_1 - x_2 \]
\[ \text{s.t.} \quad \begin{align*}
    x_1 - x_2 & \geq -1 \\
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    6x_1 + x_2 & \leq 15 \\
    x_1, x_2 & \geq 0 \\
    x_1, x_2 & \in \mathbb{Z}
\end{align*} \]

\[ d(\lambda) = \min_{x} (3x_1 - x_2 + \lambda(-1 - x_1 + x_2)) \]
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\end{align*}
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d(\lambda) = \min_{x \in X} (3x_1 - x_2 + \lambda(-1 - x_1 + x_2))
\]

\[
\Rightarrow \quad \max_{\lambda \geq 0} d(\lambda) \Rightarrow \lambda^* = \frac{5}{3}
\]

\[
\mathcal{X}(\lambda^*) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}
\]
Main Idea

The following convexification of $P$ is strongly related to $D$

$$(P_{LP}): \begin{cases} \min_{x} \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad \sum_{i \in I} H_i x_i \leq b \\ x_i \in \text{conv}(X_i) \quad \forall i \in I \end{cases}$$

Known that:
- $J^*_{P_{LP}} = J^*_D$ [Geoffrion ’74]

We show that:
1. an optimal solution $x^*_{LP}$ to $P_{LP}$ has nice properties
   - feasible w.r.t. the complicating constraints
   - returns integral solutions for $|I| - m$ subproblems

2. $x^*_{LP}$ and $x(\lambda^*)$ differ in at most $m$ subproblems

3. thus we need limited compensation to make $x(\lambda^*)$ feasible
Main Idea

The following convexification of $P$ is strongly related to $D$

$$(P_{LP}) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \leq b \\ & x_i \in \text{conv}(X_i) \quad \forall i \in I \end{cases}$$

Known that:

- $J^*_{P_{LP}} = J^*_D$ (surprising!)

We show that:

1. an optimal solution $x^*_{LP}$ to $P_{LP}$ has nice properties
   - feasible w.r.t. the complicating constraints
   - returns integral solutions for $|I| - m$ subproblems

2. $x^*_{LP}$ and $x(\lambda^*)$ differ in at most $m$ subproblems

3. thus we need limited compensation to make $x(\lambda^*)$ feasible
The Shapley-Folkman Theorem

**Theorem (Shapley-Folkman)**

Let \( S_i, \ i = 1, \ldots, |I| \) be nonempty subsets of \( \mathbb{R}^m \), with \( |I| > m \), and let \( S = S_1 + \cdots + S_{|I|} \). Then every vector \( s \in \text{conv}(S) \) can be represented as \( s = s_1 + \cdots + s_{|I|} \), where \( s_i \in \text{conv}(S_i) \) for all \( i = 1, \ldots, |I| \), and \( s_i \notin S_i \) for at most \( m \) indices \( i \).

- relatively known in economics
- Lloyd Shapley won the Nobel Prize in Economic Sciences (in '12)
- in our case

\[
S_i = H_iX_i \\
S = \bigoplus_{i \in I} S_i = \left\{ \sum_{i \in I} H_i x_i \mid x_i \in X_i \right\}
\]
Visualization of the Theorem

\[
\min_x \sum_{i \in I} c_i x_i \\
\text{s.t. } \sum_{i \in I} H_i x_i \leq b \\
x_i \in X_i \quad \forall i \in I
\]
Visualization of the Theorem

\[
\begin{align*}
\min_x & \quad \sum_{i \in I} c_i x_i \\
\text{s.t.} & \quad \sum_{i \in I} H_i x_i \leq b \\
& \quad x_i \in X_i \quad \forall i \in I
\end{align*}
\]

(a) subsystems $X_i$
(b) budget consumption $H_i X_i$
Visualization of the Theorem

\[
\min_x \quad \sum_{i \in I} c_i x_i \\
\text{s.t.} \quad \sum_{i \in I} H_i x_i \leq b \\
x_i \in X_i \quad \forall i \in I
\]

(a) subsystems \( X_i \)  \hspace{1cm} (b) budget consumption \( H_i X_i \)

(c) aggregated budget consumption
Visualization of the Theorem

\[
\begin{align*}
\min_{x} & \quad \sum_{i \in I} c_i x_i \\
\text{s.t.} & \quad \sum_{i \in I} H_i x_i \leq b \\
& \quad x_i \in \text{conv}(X_i) \quad \forall i \in I
\end{align*}
\]

(a) subsystems $X_i$

(b) budget consumption $H_i X_i$

(c) aggregated budget consumption $S$ and $\text{conv}(S)$
Visualization of the Theorem

\[
\min_x \quad \sum_{i \in I} c_i x_i \\
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x_i \in \text{conv}(X_i) \quad \forall i \in I
\]

(a) subsystems $X_i$    
(b) budget consumption $H_i X_i$    
(c) aggregated budget consumption

$S$ and $\text{conv}(S)$
Result #1: Bounded Duality Gap

- Duality gap for programs structured as \( P \) is bounded:

**Theorem (bound on duality gap (Ekeland ’76, Bertsekas ’83))**

Assume that the sets \( X_i \) are non-empty and compact, and that for any given \( x_i \in \text{conv}(X_i) \), there exists an \( \tilde{x}_i \in X_i \) such that \( H_i \tilde{x}_i \leq H_i x_i \). Then

\[
J^*_P - J^*_D \leq m \cdot \max_{i \in I} \left( \max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)
\]

- non-convexities do not compound each other indefinitely in \( P \), i.e. the distance between \( P \) and \( P_{LP} \) does not grow
- Thus, if \( J^*_P \) increases linearly with \( |I| \),

\[
\frac{J^*_P - J^*_D}{J^*_P} \to 0 \quad \text{as} \quad |I| \to \infty.
\]
Example #2: Partial Shipments

- distributor has to supply $I$ customers
- often, available inventory $< \text{total demand}$
  - demand uncertainty and high storage costs
  - restrictions on manufacturing capacity
- thus
  - \textbf{EITHER} fully supply a smaller set of customers
  - \textbf{OR} partially supply a larger set
- problem is to decide which customer gets what product in the presence of shipping restrictions
Example #2: Partial Shipments

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  - EITHER fully supply a smaller set of customers
  - OR partially supply a larger set
- problem is to decide which customer gets what product in the presence of shipping restrictions
Optimization Problem Model

\[ i = \text{customer}, \quad j = \text{product} \]
\[ w_i = \text{ship to } i \ (\text{yes/no}), \quad S_i^j = \text{fraction shipped} \]

\[
\begin{align*}
\max & \quad \sum_i \text{reward for } w_i + \sum_{i,j} \text{reward for } S_i^j \\
\text{s.t.} & \quad \sum_i S_i^j \leq \text{inventory of prod. } j \\
& \quad \begin{cases}
0 & \text{if } w_i = 0 \\
\sum_j S_i^j \geq \text{min. shipment} \quad \text{and } S_i^j \leq \text{demand of } j & \text{if } w_i = 1
\end{cases}
\end{align*}
\]

• compare with generic model \( P \):

\[
P : \left\{ \begin{array}{l}
\min_x & \sum_{i \in I} c_i x_i \\
\text{s.t.} & \sum_{i \in I} H_i x_i \leq b \\
& x_i \in X_i \quad \forall i \in I
\end{array} \right\}
\]
Results using Proposed Method

- partial shipment problem is NP-Hard
- greedy strategies perform poorly
- tailored MINTO setup also not good (see [Dawande '06])
  ▶ at $N = 300, M = 75$
    avg opt. gap = 6.2% and computation time 6h
- generate instances of industrial size
- solve with adapted method

<table>
<thead>
<tr>
<th>cust.</th>
<th>prod.</th>
<th>Proposed Method</th>
<th>CPLEX</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>gap (%) †</td>
<td>time (s) †</td>
</tr>
<tr>
<td>100</td>
<td>25</td>
<td>0.38</td>
<td>18.5</td>
</tr>
<tr>
<td>300</td>
<td>75</td>
<td>0.03</td>
<td>62.5</td>
</tr>
<tr>
<td>500</td>
<td>50</td>
<td>0.06</td>
<td>101.0</td>
</tr>
<tr>
<td>600</td>
<td>50</td>
<td>0.06</td>
<td>116.6</td>
</tr>
<tr>
<td>1000</td>
<td>100</td>
<td>0.03</td>
<td>245.2</td>
</tr>
</tbody>
</table>

† average over 10 instances
$x^*_\text{LP}$ can be computed

1. solve $\mathcal{D}$ using a subgradient method

$$
\begin{align*}
\lambda^{[1]} &= 0 \\
\lambda^{(k+1)} &= P_+ \left( \lambda^{(k)} + s^{(k)} \cdot \gamma^{(k)} \right)
\end{align*}
$$

- $s^{(k)}$ steplength, pick $s^{(k)} = \alpha/k$.
- $\gamma^{(k)}$ a subgradient, pick $\gamma^{(k)} = \sum_{i \in I} H_i x_i(\lambda^{(k)}) - b$.

2. while iterating (1), calculate average

$$
\bar{x}^{(k)} = \frac{1}{k} \sum_{j=1}^{k} x(\lambda^{(j)})
$$

### Theorem (primal + dual convergence)

- $\lambda^{(k)} \rightarrow \lambda^* \in \Lambda^*$,
- if $x^*_\text{LP}$ is the unique solution to $\mathcal{P}_\text{LP}$, $\bar{x}^{(k)} \rightarrow x^*_\text{LP}$.
Computing approximate solutions to $\mathcal{P}$

Assume $H_i x_i \geq 0$, and $0 \in X_i$ (relaxed in [Vujanic '13]).

- choices of $x_i$ always “consume” budget

Procedure to compute approx. $\hat{x}^*$ to $\mathcal{P}$ (distributed)

1. compute $x_{LP}^*$ using averaging
2. find $I_1 \subseteq I$ such that $(x_{LP}^*)_i \in \text{vert}(X_i)$
   - $|I_1| \geq |I| - m - 1$
3. for $i \in I_1$, set $\hat{x}_i^* = (x_{LP}^*)_i$
4. for the remaining $i \in I \setminus I_1$,
   - EITHER $\hat{x}_i^* = 0$
   - OR set $b \rightarrow b - \sum_{i \in I_1} H_i \hat{x}_i^*$, and solve smaller dimensional problem

Then, $\hat{x}^*$ is feasible for $\mathcal{P}$, and satisfies

$$J_\mathcal{P}(x_i^*) - J_\mathcal{P}^* \leq (m + 1) \max_{i \in I} \max_{x_i \in X_i} c_i^T x_i.$$
Control Strategy

- Optimal Control + Lagrangian Duality
- Optimization Problem Model

\[
\text{minimize} \quad \sum_{i \in I} P_i \left( C^u \cdot u_i - C^v \cdot v_i \right)
\]

subject to

\[
P^{\text{min}} \leq \sum_{i \in I} P_i (u_i - v_i) \leq P^{\text{max}}
\]

\[
(e_i, u_i, v_i) \in X_i
\]

with local (private) model

\[
X_i = \left\{ \begin{bmatrix} e_i \\ u_i \\ v_i \end{bmatrix} \in \mathbb{R}^N \times \mathbb{Z}^{2N} \left| \begin{array}{l}
E_i^{\text{init}} + B (E_i^{\text{in}} u_i - E_i^{\text{out}} v_i) \\
E_i^{\text{min}} \leq e_i \leq E_i^{\text{max}} \\
e_i[N] \geq E_i^{\text{ref}} \\
0 \leq u_i + v_i \leq 1
\end{array} \right. \right\}
\]

- \( u_i[k], v_i[k] \in \{0, 1\} \): charge/discharge decision at step \( k \) for EV \( i \in I \)
- \( P_i \): charge rate
- \( C^u[k], C^v[k] \): prices for charging/discharging
Solution Method

1. set

\[
\begin{align*}
\rho_{\text{charge}} &= N \cdot \max_{i \in I} P_i \\
\rho_{V2G} &= 2N \cdot \max_{i \in I} P_i \\
\end{align*}
\]

\[\Rightarrow \left\{ \begin{array}{c}
\bar{P}^\text{min} = P^\text{min} + \rho \\
\bar{P}^\text{max} = P^\text{max} - \rho \\
\end{array} \right. \]

2. add \( \delta_i^u[k], \delta_i^v[k] \) perturbations to cost function

3. dual (outer) problem
   - solved using a subgradient method
   - constant stepsize, decreased every 20–30 iterations

4. inner problem
   - **charge only**: optimal local solution is greedy (easy to show)
   - **charge+V2G**: deploy DP (local problems are 1D)
Results (5000 EVs) (1/2)
Results (5000 EVs) (1/2)

Iteration # 40

\[ \sum_{i \in I} p_i : \text{power flow (MW)} \]
Results (5000 EVs) (1/2)
Results (5000 EVs) (1/2)
Results (5000 EVs) (1/2)
Results (5000 EVs) (2/2)

(c) with disturbances

(d) without disturbances
## Results: Charge Only

<table>
<thead>
<tr>
<th># of EVs</th>
<th>Proposal Method</th>
<th>CPLEX</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Opt. Gap (%)</td>
<td>Solve time (sec)</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>Avg</td>
</tr>
<tr>
<td>200</td>
<td>3.24</td>
<td>3.32</td>
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<tr>
<td>350</td>
<td>2.21</td>
<td>2.44</td>
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<tr>
<td>500</td>
<td>1.40</td>
<td>1.46</td>
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<tr>
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<td>1.01</td>
<td>1.05</td>
</tr>
<tr>
<td>1000</td>
<td>0.68</td>
<td>0.72</td>
</tr>
<tr>
<td>1500</td>
<td>0.46</td>
<td>0.47</td>
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<tr>
<td>2000</td>
<td>0.33</td>
<td>0.35</td>
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<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>10000</td>
<td>0.03</td>
<td>0.03</td>
</tr>
</tbody>
</table>

(*) \(\leq 0.3\) sec (imprecise measurements).

- all solutions recovered are feasible
- fast computation times due to greedy subproblem structure
## Results: Charge + V2G

<table>
<thead>
<tr>
<th># of EVs</th>
<th>Proposed Method</th>
<th>CPLEX</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Opt. Gap (%)</td>
<td>Solve time (min)</td>
<td>Solve time (min)</td>
</tr>
<tr>
<td></td>
<td>Min</td>
<td>Avg</td>
<td>Max</td>
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<td>3.51</td>
</tr>
<tr>
<td>500</td>
<td>2.05</td>
<td>2.15</td>
<td>2.24</td>
</tr>
<tr>
<td>700</td>
<td>1.48</td>
<td>1.54</td>
<td>1.61</td>
</tr>
<tr>
<td>1000</td>
<td>1.01</td>
<td>1.05</td>
<td>1.10</td>
</tr>
<tr>
<td>1500</td>
<td>0.65</td>
<td>0.68</td>
<td>0.72</td>
</tr>
<tr>
<td>2000</td>
<td>0.45</td>
<td>0.50</td>
<td>0.53</td>
</tr>
<tr>
<td>5000</td>
<td>0.12</td>
<td>0.15</td>
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<tr>
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<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td>10000</td>
<td>0.06</td>
<td>0.07</td>
<td>0.07</td>
</tr>
</tbody>
</table>

(*) failed to solve two instances (out of memory)
(–) out of memory before attaining the desired optimality gap

- 500 EVs: CPLEX times vary from 15 min to 4.5h (6+ when unsolved)
- 10k EVs: with 500k binaries and 250k continuous vars, these are among the largest MIPs publicly available
- CPLEX does not even finish preprocessing on those
Solutions Performance

“1/|I|” rate of decrease of optimality gap verified in both cases
Intuitive explanation of proof that 1) $x_{LP}^*$ is structured and 2) that $x_{LP}^*$ and $x_i(\lambda)$ are connected

- $P_{LP}$ is equivalent to

$$\begin{cases}
\min \sum_i c_i x_i \\
\text{s.t. } \sum_i H_i x_i \leq b \quad \Leftrightarrow \quad \\
\quad x_i \in \text{conv}(X_i)
\end{cases} \quad \iff \quad 
\begin{cases}
\min_p \quad \sum_{i \in I} \sum_{j \in J_i} p_{ij} (c_i x_j^i) \\
\text{s.t. } \quad \sum_{i \in I} \sum_{j \in J_i} p_{ij} (H_i x_j^i) \leq b \\
\quad \sum_{j \in J_i} p^j_i = 1 \\
\quad p^j_i \geq 0
\end{cases}$$

- at most $|I| + m$ entries of $p$ are $> 0$
  - $p$ is dimension $K$ (huge), hence requires $K$ equalities to be determined
  - in $P_{LP}$, only $|I| + m$ equalities, so $K - (|I| + m)$ equalities have to be picked form $p^j_i \geq 0$
  - hence at most $|I| + m$ entries $p^j_i$ can be $> 0$ (“at most” comes from slack for coupling constr.)
- for at least $|I| - m$ subproblems $(p^*)^j_i = 1$
- inner problem can return $x_j^i$ only if its “probability” is $(p^*)^j_i > 0$
- we show this using strict complementarity, which requires uniqueness
Robust Schedules

Is robustification of the schedule equivalent to, e.g., increasing task length?
Robust Schedules

Is robustification of the schedule equivalent to, e.g., increasing task length?

encode possible delay of 1 unit by assuming task A is length 2
Robust Schedules

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Robust Schedules

Is robustification of the schedule equivalent to, e.g., increasing task length?

encode possible delay of 1 unit by assuming task A is length 2

the algorithm produces...
Lagrangian Duality → Divide & Conquer

- use duality to decompose $\mathcal{P}$ in smaller problems

\[
\begin{array}{ccccccc}
  c_1 & c_2 & \cdots & \cdots & \cdots & c_I \\
  H_1 & H_2 & \cdots & \cdots & \cdots & H_I \\
  A_1 & d_1 & \cdots & \cdots & \cdots & A_I \\
  A_2 & \cdots & \cdots & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  A_I & d_I & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

- but...

\(x^D\) is generally suboptimal or even infeasible in the MIP case

- solutions may violate the relaxed constraints $\sum_i H_i x_i \leq b$
Robust PWM

ROBUST OPEN LOOP/CLOSED LOOP
Linear Robust MPC

Finite horizon, optimal control problem formulation

\[
\begin{align*}
\min & \quad \left[ \sum_{k=0}^{N-1} (x_k - x_{ref})^T P (x_k - x_{ref}) + u_k^T Q u_k \right] \\
\text{s.t.} & \quad x_0 = \bar{x}_0, \\
& \quad x_{k+1} = Ax_k + Bu_k \\
& \quad x_k \in \mathcal{X}_k, \\
& \quad u_k \in \mathcal{U}_k
\end{align*}
\]

- ideally, obtain **optimal control policy** using e.g. DP
  - ”optimal decision is taken at each stage in the horizon”
  - \( u_k = \pi(x_k) \)
  - often intractable
- conservative approximation: **open-loop control policy**
  - ”decide now for the entire horizon” (plan can’t be modified)
  - \( u_k = v_k \)
  - poor performance, **infeasibility issues**
- middle ground: **affine recourse**
  - ”decisions are affinely adjusted once disturb. are meas.”
Linear Robust MPC

Finite horizon, robust optimal control problem formulation

\[
\min \mathbb{E} \left[ \sum_{k=0}^{N-1} (x_k - x_{\text{ref}})^T P (x_k - x_{\text{ref}}) + u_k^T Q u_k \right]
\]

s.t.  
\[
x_0 = \bar{x}_0, \quad x_{k+1} = A x_k + B u_k + G w_k, \quad x_k \in X_k, \quad u_k \in U_k
\]

\forall w \in W

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  - ”decisions are affinely adjusted once disturb. are meas.”
Linear Robust MPC

Finite horizon, optimal control problem formulation

\[
\begin{align*}
\min \quad & \mathbb{E} \left[ (x - x_{\text{ref}})^T P (x - x_{\text{ref}}) + u^T Qu \right] \\
\text{s.t.} \quad & x = Ax_0 + Bu + Gw, \\
& \mathbb{E} x x + \mathbb{E} u u \leq e \\
& \forall w \in \mathcal{W}
\end{align*}
\]

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Linear Robust MPC

Finite horizon, optimal control problem formulation

\[
\begin{align*}
\min & \quad \mathbb{E} \left[ (x - x_{\text{ref}})^T P (x - x_{\text{ref}}) + u^T Qu \right] \\
\text{s.t.} & \quad x = Ax_0 + Bu + Gw, \\
& \quad E_x x + E_u u \leq e \\
& \quad \forall w \in \mathcal{W}
\end{align*}
\]

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  - "optimal decision is taken at each stage in the horizon"
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  - poor performance, infeasibility issues

- middle ground: **affine recourse**
  - "decisions are affinely adjusted once disturb. are meas."
  - \( u_k = v_k + \sum_{j=0}^{k} K_{kj} x_j \)
Linear Robust MPC

Finite horizon, optimal control problem formulation

\[
\min \mathbb{E} \left[ (x - x_{ref})^T P (x - x_{ref}) + u^T Q u \right]
\]

s.t.

\[
x = A x_0 + B u + G w, \quad E_x x + E_u u \leq e \quad \forall w \in \mathcal{W}
\]

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  - poor performance, infeasibility issues

- middle ground: **affine recourse**
  - "decisions are affinely adjusted once disturb. are meas."
  - \( u_k = v_k + \sum_{j=0}^k M_{kj} w_j \)
Explicit Robust Counterpart

- can show that the robust counterpart is equivalent to

\[
\min_{v,M,\Lambda} \mathbb{E} \left[ (x - x_{\text{ref}})^T P (x - x_{\text{ref}}) + u^T Q u \right] \\
\text{s.t.} \quad E_x A x_0 + (E_x B + E_u) v - e \leq -\Lambda^T \cdot h \\
\Lambda^T \cdot S = E_x G + E_x B M + E_u M \\
\Lambda \geq 0
\]

where \( w_k \) is bounded by \( S \cdot w_k \leq h \) (Goulart 2006)

- finite convex QP
Extension to Hybrid Systems
Robust Control of Hybrid Systems

- hybrid system dynamics (MLD)

\[ x_{k+1} = Ax_k + Bu_k + B_2 \delta_k + B_3 z_k + Gw_k \]
\[ E_x x_k + E_u u_k + E_\delta \delta_k + E_z z_k \leq e_k \]
\[ \delta_k \in \{0, 1\}^{n_\delta}, z_k \in \mathbb{R}^{n_z} \]

- \( \delta_k, z_k \) characterize hybrid behavior,
  - in the dynamics, e.g. switching between modes
  - in the constr., e.g. logic conditions on the inputs

- wish to obtain a solution (with some recourse) to

\[
\min_{x,u,\delta,z} \mathbb{E} \left[ (x - x_{\text{ref}})^T P(x - x_{\text{ref}}) + u^T Qu \right]
\]
\[
\text{s.t. } x = Ax_0 + Bu + B_2 \delta + B_3 z + Gw,
E_x x + E_u u + E_\delta \delta + E_z z \leq e, \quad \forall w \in \mathcal{W}
\]
\[ \delta \in \{0, 1\}^{N \cdot n_\delta} \]

(RHOCPS)
**Affine recourse on continuous inputs - R-MPC$_{hyb}$**

**Main Idea**

Proposed idea:
- we need recourse, but affine functions cannot easily provide binary inputs
- split the inputs into continuous inputs $u$ and binary inputs $d$

$$x_{k+1} = A \cdot x_k + B_{cont} u_k + B_{bin} d_k + B_2 \cdot \delta_k + B_3 z_k + Gw_k$$

- introduce affine recourse on the continuous inputs

$$u := M \cdot w + v$$

**Assumption on $G$:**
- disturbances only affect the continuous dynamics
can show that the robust counterpart is

\[
\min_{v, M, d, \delta, z, \Lambda} \hat{f} + \text{trace}(D \cdot C_w)
\]

s.t.
\[
\hat{e} \leq -\Lambda^T h,
\]
\[
\Lambda^T S = E_x B_u M + E_x G + E_u M
\]
\[
\Lambda \geq 0 \quad \text{element-wise}
\]
\[
\delta \in \{0, 1\}^{N \cdot n_\delta}, \quad d \in \{0, 1\}^{N \cdot n_d}
\]

with
\[
\hat{x} \doteq A x_0 + B_u v + B_d d + B_2 \delta + B_3 z
\]
\[
\hat{e} \doteq E_x \hat{x} + E_u v + E_d d + E_\delta \delta + E_z z - e
\]

(variables in case of zero disturbance)

and appropriate \(D\)
Example: DC-DC Buck Converter

Linear Model
Plant: the Buck Converter (BC) (1/2)

- BC regulates input voltage $V_{in}$ down to desired $V_{o,ref}$
- operated by switch (controlled input)
- disturbances: $|i_d| \leq 0.5 \cdot i_{L,ref}$
- state constraints: $i_L \leq 2 \cdot i_{L,ref}$
- sampling frequency: 10 kHz

Figure: DC-DC buck converter circuit
Averaged Model (standard method)

- replace $\delta(t) \in \{0, 1\}$ by duty cycle $d(t) \in [0, 1]$
- average dynamics "when off" and "when on" weighted by $d(t)$, obtain

$$\dot{x} = Ax + Bu + Gw$$

$u \in [0, 1]$

- linear model
Performance of R-MPC$_{avg}$
R-MPC_{avg} – Second Experiment
Hybrid Model [2] (1/2)

- Closer approximation of the switch
- Divide time into cycles with M samples each
- New binary inputs: switch on $\delta_k^+ = 1$, switch off $\delta_k^- = 1$
- New continuous input: $u_{c,k}$
- New auxiliary state: $s_k$ (integration of $\delta^+ - \delta^-$)
- Switch position given by $s_k + u_{c,k}$

Figure: PWM in the hybrid model
Hybrid Model [2] (2/2)

- New system equations:

\[
\begin{pmatrix}
  i_L \\
  V_C \\
  s
\end{pmatrix}_{k+1} =
\begin{pmatrix}
  A_{11} & A_{12} & B_1 \\
  A_{21} & A_{22} & B_2 \\
  0 & 0 & 1
\end{pmatrix} \cdot
\begin{pmatrix}
  i_L \\
  V_C \\
  s
\end{pmatrix}_k
+ \begin{pmatrix}
  B_1 & 0 & 0 \\
  B_2 & 0 & 0 \\
  0 & 1 & -1
\end{pmatrix} \cdot
\begin{pmatrix}
  u_c \\
  \delta_+ \\
  \delta_-
\end{pmatrix}_k
+ \begin{pmatrix}
  G_1 \\
  G_2 \\
  0
\end{pmatrix} w_k
\]

- New constraints:

\[
\begin{align*}
\delta^+_k, \delta^-_k & \in \{0, 1\} & \text{binary inputs} \\
0 & \leq s_k \leq 1 & \text{binary state} \\
0 & \leq s_k + u_{c,k} \leq 1 & \text{limited input} \\
-\delta^-_k & \leq u_{c,k} \leq \delta^+_k & \text{switching time} \\
\sum_{i=0}^{M-1} \delta^+_k & \leq 1 & \text{switching constraints} \\
\sum_{i=0}^{M-1} \delta^-_k & \leq 1 \\
\end{align*}
\]

- together with \( i_{L,k} \leq 2 \cdot i_{L,\text{ref}} \) they form

\[
\begin{align*}
x_{k+1} &= A \cdot x_k + B_{\text{cont}} u_k + B_{\text{bin}} d_k + B_2 \cdot \delta_k + B_3 z_k + G w_k \\
E_x x_k + E_u u_k + E_d d_k + E_\delta \delta_k + E_z z_k & \leq e_k
\end{align*}
\]
Performance & Simulation results – R-MPC$_{hyb}$

![Graph showing simulation results](image-url)
R-MPC$_{hyb}$ – Second Experiment

<table>
<thead>
<tr>
<th>Controller</th>
<th>RMS deviation of $V_o(t)$</th>
<th>RMS deviation of $V_o(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1^{st}$ experiment</td>
<td>$2^{nd}$ experiment</td>
</tr>
<tr>
<td>$MPC_{avg}$</td>
<td>2.09[V]</td>
<td>15.2[V]</td>
</tr>
<tr>
<td>$MPC_{hyb}$</td>
<td>0.88[V]</td>
<td>9.74[V]</td>
</tr>
<tr>
<td>$MPC_{hyb}^{open\ loop}$</td>
<td>0.89[V]</td>
<td>Infeasibility encountered</td>
</tr>
</tbody>
</table>