Large Scale Mixed-Integer Optimization: a Solution Method with Supply Chain Applications

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Outline

Large MIPs: Problems' Structure

We investigate large scale problems in the form:

$$(P) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \leq b \\ & x_i \in X_i \qquad\qquad \forall i \in I, \end{cases}$$

with X_i compact mixed integer polyhedral sets

$$X_i = \left\{ x \in \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i} \;\middle|\; A_i x \leq d_i \right\}.$$

Interested in $|I| \gg$ coupling constraints.

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Application Example: Partial Shipments

- distributor has to supply M products to N customers
  - usually, $N \gg M$
- often, avail. inventory $<$ total demand
  - demand uncertainty and high storage costs
  - restrictions on manufacturing capacity
- thus
  - **EITHER** fully supply a smaller set of customers
  - **OR** partially supply a larger set
- but shipments cannot be too small

Partial Shipments Problem [Dawande '06]

Allocate products to customers in the presence of shipping restrictions.

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Partial Shipments Problem [Dawande '06]

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Optimization Problem Model (1/2) [Dawande '06]

$$\begin{array}{ll} i = \text{customer}, & \text{j} = \text{product} \\ w_i = \text{ship to i (yes/no)}, & S_i^j = \text{fraction shipped} \end{array}$$

$$\left\{ \begin{array}{lll} \max & \sum\limits_i \text{reward for } w_i + \sum\limits_{i,j} \text{reward for } S_i^j & \\ \text{s.t.} & \sum\limits_i S_i^j \leq \text{inventory budget of prod. j} & \\ & x_i = \begin{bmatrix} w_i \\ S_i^j \end{bmatrix} \in \left\{ \begin{array}{lll} \text{if } w_i = 1 & \Rightarrow & \sum_j S_i^j \geq \text{min. shipment} \\ & & \text{and } S_i^j \leq \text{demand of } j \\ \text{if } w_i = 0 & \Rightarrow & S_i^j = 0 \\ w_i \in \{0, 1\} & & \end{array} \right\} & \end{array} \right.$$

Compare with generic model P:

$$\left\{ \begin{array}{lll} \min_x & \sum_{i \in I} c_i x_i & \\ \text{s.t.} & \sum_{i \in I} H_i x_i \leq b & \\ & x_i \in X_i & \forall i \in I, \end{array} \right.$$

More examples

- power systems
  - (CDC) electric vehicles charging coordination
  - demand side management of TCLs
  - scheduling a *large* fleet of generators
- many traditional combinatorial problems
  - (multidimendional) knapsack problems
  - generalized assignments
  - ...
- portfolio models for small investors
- automatic reformulations
- maybe your application?

Outline

Problem's Decomposition

Obtain decomposition using duality:

$$\begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & x_i \in X_i \end{array} \Rightarrow \underbrace{\sum_{i \in I} \left\{ \min_{x_i \in X_i} c_i x_i + \lambda'(H_i x_i) \right\} - \lambda' b}_{\doteq d(\lambda)}$$

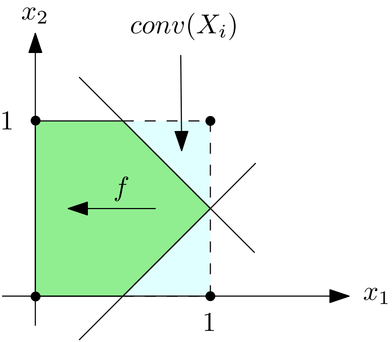
- Lagrangian dual (or outer) problem:

$$(D) : \left\{ \begin{array}{ll} \sup_\lambda & d(\lambda) \\ \text{s.t.} & \lambda \succeq 0 \end{array} \right.$$

Consider solutions to **inner problem** as candidate solutions for P

$$x_i(\lambda) \in \arg\min_{x_i \in X_i} \left\{ (c_i + \lambda' H_i) x_i \right\} \quad \lambda \in \mathbb{R}^m_+.$$

Issues affecting inner solutions



$$
\begin{array}{ll}
\min & -x_1 \\
\text{s.t.} & x_1 - x_2 \leq 0.5 \\
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 & x_1, x_2 \in \{0, 1\}
\end{array}
$$

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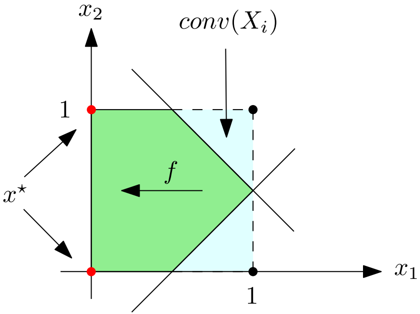

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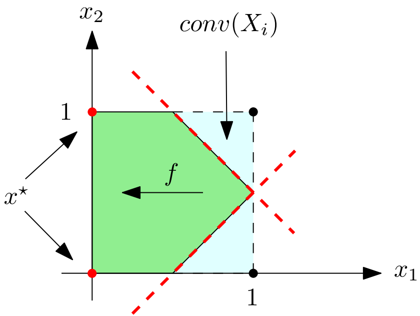

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$$\Rightarrow \max_{\lambda \geq 0} \min_{x_1, x_2 \in \{0,1\}} \quad -x_1 + \lambda_1 \underline{(x_1 - x_2 - 0.5)} + \lambda_2 \underline{(x_1 + x_2 - 1.5)}$$

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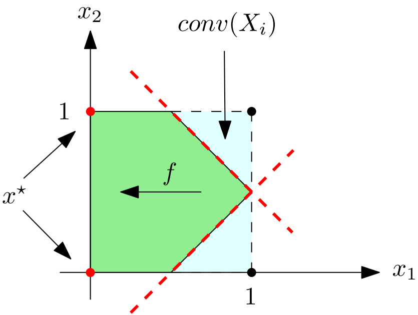

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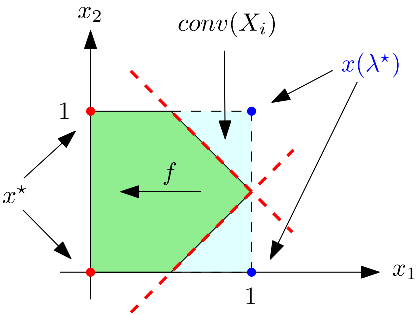
$$\Rightarrow d^\star = -1 \qquad \lambda^\star = (0.25, 0.25)$$

Issues affecting inner solutions



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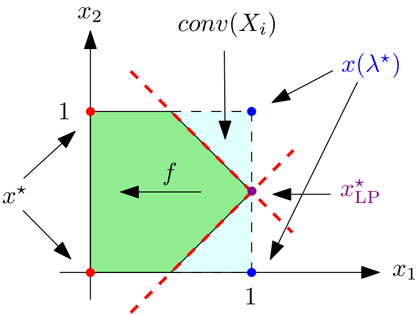

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Main Idea

The following convexification of P is strongly related to D

$$(P_{\mathrm{LP}}) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & x_i \in \mathrm{conv}(X_i) \quad \forall i \in I \end{cases}$$

Known that:

- $J^\star_{P_{LP}} = J^\star_D$ (**surprising!**)
- $J^\star_D \to J^\star_P$ as $|I| \to \infty$, vanishing duality gap [Aubin '76]

We show

1. **an optimal solution $x^\star_{\mathrm{LP}}$ to P_{LP} is useful**
   - feasible w.r.t. the complicating constraints
   - $x^\star_{\mathrm{LP}}$ is non-integer in at most m subsystems
   - Shapley–Folkman–Starr Theorem
2. **we can compute it**, borrowing methods from conv. opt.

$x^\star_{\mathrm{LP}}$ can be computed

1. solve (D) using a subgradient method

$$\begin{aligned} \lambda^{[1]} &= 0 \\ \lambda^{[k+1]} &= \mathrm{P}_+\left(\lambda^{[k]} + s^{[k]} \cdot \gamma^{[k]}\right) \end{aligned} \qquad (1)$$

   - $s^{[k]}$ steplength, pick $s^{[k]} = \alpha/k$.
   - $\gamma^{[k]}$ a subgradient, pick $\gamma^{[k]} = \sum_{i\in I} H_i x_i(\lambda^{[k]}) - b$.

2. while iterating (1), calculate average

$$\bar{x}^{[k]} = \frac{1}{k}\sum_{j=1}^{k} x(\lambda^{[j]})$$

Theorem (primal + dual convergence)

- $\lambda^{[k]} \to \lambda^\star \in \Lambda^\star$,
- *if $x^\star_{\mathrm{LP}}$ is the unique solution to $\mathcal{P}_{\mathrm{LP}}$, $\bar{x}^{[k]} \to x^\star_{\mathrm{LP}}$.*

Computing approximate solutions to (P)

Assume $H_i x_i \geq 0$, and $0 \in X_i$ (relaxed in [Vujanic '13]).

- choices of x_i always "consume" budget

Procedure to compute approx. $\hat{x}^\star$ to P (distributed)

1. compute $x^\star_{\mathrm{LP}}$ using averaging
2. find $I_1 \subseteq I$ such that $(x^\star_{\mathrm{LP}})_i \in \mathrm{vert}(X_i)$
   - $|I_1| \geq |I| - m - 1$
3. for $i \in I_1$, set $\hat{x}^\star_i = (x^\star_{\mathrm{LP}})_i$
4. for the remaining $i \in I \setminus I_1$,
   - EITHER $\hat{x}^\star_i = 0$
   - OR set $b \to b - \sum_{i \in I_1} H_i \hat{x}^\star_i$, and solve smaller dimensional problem

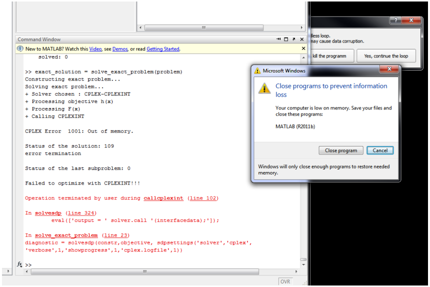
Then, $\hat{x}^\star$ is feasible for P, and satisfies

$$J_{\mathcal{P}}(\hat{x}^\star) - J^\star_{\mathcal{P}} \leq (m+1) \max_{i \in I} \max_{x_i \in X_i} c_i^\top x_i .$$

Outline

Partial Shipment is NP-Hard

- partial shipment problem is NP-Hard (reduction to knapsack)



CPLEX with "just" $N = 300$, $M = 75$

- greedy strategies perform poorly
- tailored MINTO setup also not good (see [Dawande '06])
  - at $N = 300$, $M = 75$
    avg opt. gap $= 6.2\%$ and computation time 6h

Results using Proposed Method

- generate instances of industrial size, data from [Dawande]

		Proposed Method		CPLEX
N	M	Gap (%)	Time (s)	Time (s)
100	25	0.11–0.38–0.60	16.7–18.5–21.5	10.3–63.5–202.5
300	75	0.01–0.03–0.07	61.0–62.5–63.4	–
500	50	0.04–0.06–0.11	95.0–101.0–110.6	–
600	50	0.01–0.06–0.10	113.7–116.6–119.5	–
1000	100	0.00–0.03–0.05	225.3–245.2–288.8	–

- works well because the integer part converges very quickly

Thank you for your attention!

Questions?

References:

- R. Vujanic, P. Mohajerin Esfahani, P. Goulart, S. Mariethoz and M. Morari, *Vanishing Duality Gap in Large Scale Mixed-Integer Optimization: a Solution Method with Power System Applications*, submitted to Mathematical Programming (2013).
- J. P. Aubin and I. Ekeland, *Estimates of the duality gap in nonconvex optimization*, Mathematics of Operations Research **1** (1976), no. 3, 225-–245.
- Dimitri P. Bertsekas, G. Lauer, N. Sandell, and T. Posbergh, *Optimal short-term scheduling of large-scale power systems*, IEEE Transactions on Automatic Control **28** (1983), no. 1, 1– 11.
- Milind Dawande, Srinagesh Gavirneni, and Sridhar Tayur, *Effective heuristics for multiproduct partial shipment models*, Operations Research **54** (2006), no. 2, 337–352 (en).

Bkacup slides

BACKUP SLIDES

Shapley Folkman Theorem

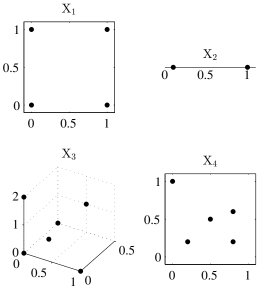
Theorem (, '69)

Let S_i, $i = 1, \ldots, |I|$ be nonempty subsets of $\mathbb{R}^m$, with $|I| > m$, and let $S = S_1 + \cdots + S_{|I|}$. Then every vector $s \in conv(S)$ can be represented as $s = s_1 + \cdots + s_{|I|}$, where $s_i \in conv(S_i)$ for all $i = 1, \ldots, |I|$, and $s_i \notin S_i$ for at most m indices i.

1) $x^{\star}_{\mathrm{LP}}$ is useful

Visualization of the Theorem

$$\begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & x_i \in X_i \qquad\qquad \forall i \in I \end{cases}$$



(a) subsystems X_i

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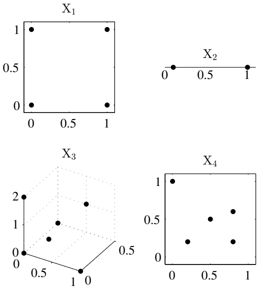
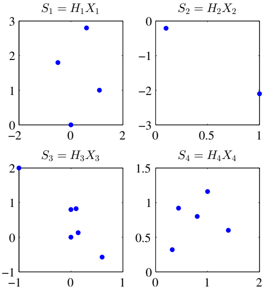


(d) subsystems X_i



(e) budget consumption $H_i X_i$

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Visualization of the Theorem

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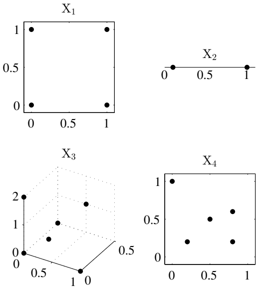
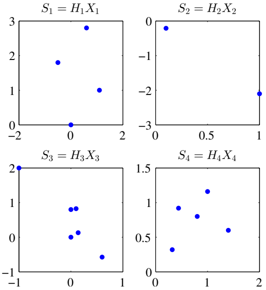
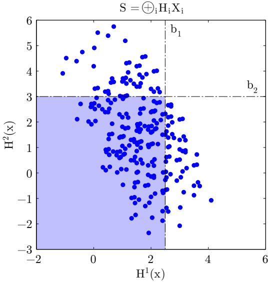


(g) subsystems X_i



(h) budget consumption $H_i X_i$

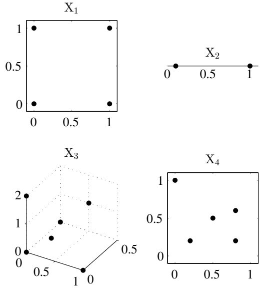


(i) aggregated budget consumption

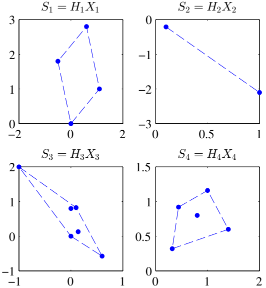
1) $x^\star_{\mathrm{LP}}$ is useful

Visualization of the Theorem

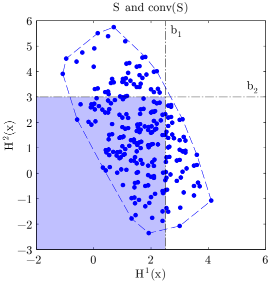
$$\begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & x_i \in \color{red}{conv}(X_i) \quad \forall i \in I \end{cases}$$



(j) subsystems X_i



(k) budget consumption $H_i X_i$



(l) aggregated budget consumption

1) $x^{\star}_{\mathrm{LP}}$ is useful

Visualization of the Theorem

$$
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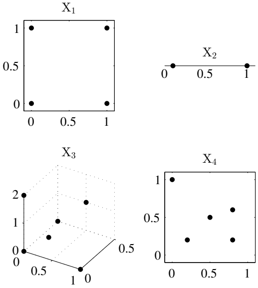
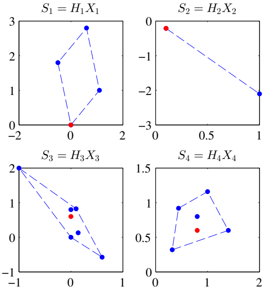
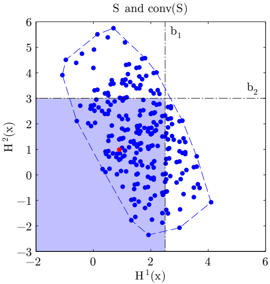
$$



(m) subsystems X_i



(n) budget consumption $H_i X_i$



(o) aggregated budget consumption

Result #1: Vanishing Duality Gap

- Duality for programs structured as (3) is **almost** strong:

Theorem (bound on duality gap (Ekeland '76, Bertsekas '83))

Assume that the sets X_i are non-empty and compact, and that for any given $x_i \in \mathrm{conv}(X_i)$, there exists an $\tilde{x}_i \in X_i$ such that $H_i \tilde{x}_i \preceq H_i x_i$. Then

$$J^\star_{\mathcal{P}} - J^\star_{\mathcal{D}} \leq m \cdot \max_{i \in I} \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)$$

- non-convexities do not compound each other indefinitely in P, i.e. the distance between P and P_{LP} does not grow
- Thus, if $J^\star_{\mathcal{P}}$ increases linearly with $|I|$,

$$\frac{J^\star_{\mathcal{P}} - J^\star_{\mathcal{D}}}{J^\star_{\mathcal{P}}} \to 0 \quad \text{as} \quad |I| \to \infty.$$

Result #2: Primal Recovery Scheme

- immunize 3 against disturbances of m subsystems:

$$(\overline{P}):\begin{cases}\min_x & \sum_{i\in I} c_i x_i \\ \text{s.t.} & \sum_{i\in I} H_i x_i \preceq \bar{b} \\ & x_i \in X_i \qquad\qquad \forall i \in I,\end{cases}$$

with $\bar{b} = b - \rho$ and

$$\rho^k = m \cdot \max_{i\in I}\left(\max_{x_i\in X_i} H_i^k x_i - \min_{x_i\in X_i} H_i^k x_i\right)$$

Theorem (feasible solutions)

If $\overline{P}_{\mathrm{LP}}$ and $\overline{D}$ have unique solutions $\bar{x}^\star_{\mathrm{LP}}$ and $\bar{\lambda}^\star$, then any selection $x(\bar{\lambda}^\star) \in \mathcal{X}(\bar{\lambda}^\star)$ is feasible for 3.

- $\bar{\lambda}^\star \succeq \lambda^\star \rightarrow$ it is like imposing a "price uplift"
- conservatism reduction possible in some cases

Performance of the Recovered Solutions

- under some technical assumption, ...

Theorem (solutions performance)

The recovered solution $x(\bar{\lambda}^\star)$ is feasible and satisfies

$$J_{\mathcal{P}}(x(\bar{\lambda}^\star)) - J_{\mathcal{P}}^\star \leq (m + ||\rho||_\infty / \zeta) \cdot \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)$$

- if $J_{\mathcal{P}}^\star$ grows linearly with $|I|$, and X_i uniformly bounded

$$\frac{J(x(\bar{\lambda}^\star)) - J_{\mathcal{P}}^\star}{J_{\mathcal{P}}^\star} \to 0 \quad \text{as} \quad |I| \to \infty,$$

- in fact, optimality gap decreases at a "$1/|I|$" rate

Intuitive explanation of proof that $x^\star_{\mathrm{LP}}$ and $x_i(\lambda)$ are connected

- Problem $\mathcal{P}_{\mathrm{LP}}$ is equivalent to

$$
\begin{array}{lll}
\min_p & \sum_{i\in I}\sum_{j\in J_i} p_{ij}(c_i x_i^j) & \\
\text{s.t.} & \sum_{i\in I}\sum_{j\in J_i} p_{ij}(H_i x_i^j) \leq b & \\
 & \sum_{j\in J_i} p_i^j = 1 & \forall i \in I \\
 & p_i^j \geq 0 & \forall i \in I, j \in J_i,
\end{array}
\qquad (\overline{P})
$$

- Inner problems can return solution x_i^j only if its "probability" is $(p^\star)_i^j > 0$.
- For $|I| - m$ subproblems only one vertex $\hat{\jmath}_i$ has probability > 0 (in fact, $(p^\star)_i^{\hat{\jmath}_i} = 1$), so they must be picked.
- Proof uses strict complementarity, which **requires uniqueness**.

Uniqueness assumption

"Nice" assumption

$x^\star_{\mathrm{LP}}$ and $\lambda^\star$ are the unique optimal solutions to $\mathcal{P}_{\mathrm{LP}}$ and (D)

- sufficient but *not necessary*
- it is **stable** (for given P, it holds with probability 1)
- and **sometimes easy to check** with standard LP tools
  - e.g. problems with integrality property, such as KPs and CLPs
- adding δ_i perturbations to cost function to ensure it **substantially enhances dual convergence**
  - $||\delta_i||_2$ 2-3 orders of magnitude smaller than $||c_i||_2$ is sufficient

Detour: Strength of Relaxations (1/2)

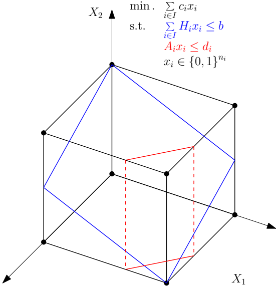
- Assume pure binary P
- If you had to solve it, which convexification would you pick?

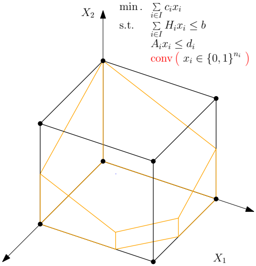
$$P_1: \quad \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & A_i x_i \preceq d_i \\ & \text{conv}(x_i \in \{0,1\}^{n_i}) \end{array} \qquad P_2: \quad \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & \text{conv}\left(\begin{array}{l} A_i x_i \preceq d_i \\ x_i \in \{0,1\}^{n_i} \end{array} \right) \end{array}$$

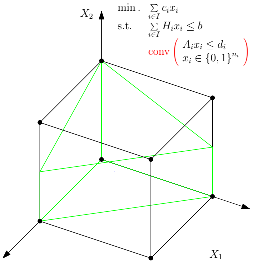
or

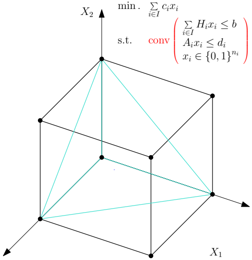
$$P_3: \quad \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \text{conv}\left(\begin{array}{l} \sum_{i \in I} H_i x_i \preceq b \\ A_i x_i \preceq d_i \\ x_i \in \{0,1\}^{n_i} \end{array} \right) \end{array}$$

Detour: Strength of Relaxations (2/2)









Performance Guarantees

- Existence and uniqueness of $\bar{x}^\star_{\mathrm{LP}}$ and $\bar{\lambda}^\star$
- There exist $\zeta > 0$ and $\hat{x}_i \in \mathrm{conv}(X_i)$ for all $i \in I$ such that

$$\sum_{i \in I} H_i \hat{x}_i \preceq b - \zeta |I| \mathbb{1}.$$

Then

$$J_{\mathcal{P}}(x(\bar{\lambda}^\star)) - J^\star_{\mathcal{P}} \leq (m + ||\rho||_\infty / \zeta) \cdot \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right). \tag{2}$$

This implies that, if $J^\star_{\mathcal{P}}$ grows linearly with $|I|$,

$$\frac{J(x(\bar{\lambda}^\star)) - J^\star_{\mathcal{P}}}{J^\star_{\mathcal{P}}} \to 0 \quad \text{as} \quad |I| \to \infty, \tag{3}$$

i.e. optimality gap decreases at a "$1/|I|$" rate.

Dual convergence

- dual convergence ($d^{(k)} \to d^\star$) is *fast*
- in all instances we tried, well within 30 dual iterations

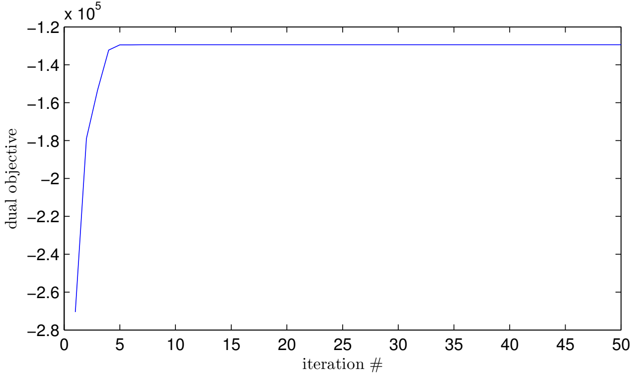


Figure: dual convergence

Ergodic Sequence: Convergence

- ergodic sequence does converge, but rather slowly
- but we don't care: we use it only to detect fractional parts

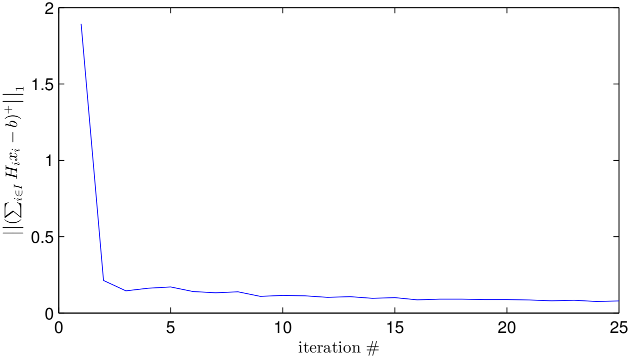


Figure: feasibility violation of ergodic sequence