

Large Scale Mixed-Integer Optimization: a Solution Method with Supply Chain Applications

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Outline

- 1 Introduction
- 2 Duality for P and Proposed Method
- 3 Results

Large MIPs: Problems' Structure

We investigate large scale problems in the form:

$$(P) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \leq b \\ & x_i \in X_i \end{cases} \quad \forall i \in I,$$

with X_i compact mixed integer polyhedral sets

$$X_i = \{x \in \mathbb{R}^{r_i} \times \mathbb{Z}^{z_i} \mid A_i x \leq d_i\}.$$

Interested in $|I| \gg$ coupling constraints.

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Application Example: Partial Shipments

- distributor has to supply M products to N customers
 - ▶ usually, $N \gg M$
- often, avail. inventory $<$ total demand
 - ▶ demand uncertainty and high storage costs
 - ▶ restrictions on manufacturing capacity
- thus
 - ▶ **EITHER** fully supply a smaller set of customers
 - ▶ **OR** partially supply a larger set
- but shipments cannot be too small



Partial Shipments Problem [Dawande '06]

Allocate products to customers in the presence of shipping restrictions.

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Partial Shipments Problem [Dawande '06]

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Optimization Problem Model (1/2) [Dawande '06]

i = customer, j = product
 w_i = ship to i (yes/no), S_i^j = fraction shipped

$$\left\{ \begin{array}{l} \max \quad \sum_i \text{reward for } w_i + \sum_{i,j} \text{reward for } S_i^j \\ \text{s.t.} \quad \sum_i S_i^j \leq \text{inventory budget of prod. } j \\ \\ x_i = \begin{bmatrix} w_i \\ S_i^j \end{bmatrix} \in \left\{ \begin{array}{l} \text{if } w_i = 1 \quad \Rightarrow \quad \sum_j S_i^j \geq \text{min. shipment} \\ \quad \quad \quad \text{and } S_i^j \leq \text{demand of } j \\ \text{if } w_i = 0 \quad \Rightarrow \quad S_i^j = 0 \\ w_i \in \{0, 1\} \end{array} \right\} \end{array} \right.$$

Compare with generic model P:

$$\left\{ \begin{array}{l} \min_x \quad \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad \sum_{i \in I} h_i x_i \leq b \\ x_i \in X_i \quad \forall i \in I, \end{array} \right.$$

More examples

- power systems
 - ▶ (CDC) electric vehicles charging coordination
 - ▶ demand side management of TCLs
 - ▶ scheduling a *large* fleet of generators
- many traditional combinatorial problems
 - ▶ (multidimensional) knapsack problems
 - ▶ generalized assignments
 - ▶ ...
- portfolio models for small investors
- automatic reformulations
- maybe your application?

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Problem's Decomposition

Obtain decomposition using duality:

$$\begin{array}{l} \min_x \quad \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad \sum_{i \in I} H_i x_i \preceq b \\ \quad \quad x_i \in X_i \end{array} \Rightarrow \underbrace{\sum_{i \in I} \left\{ \min_{x_i \in X_i} c_i x_i + \lambda'(H_i x_i) \right\}}_{\doteq d(\lambda)} - \lambda'b$$

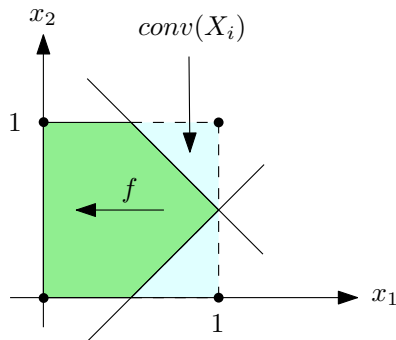
- Lagrangian dual (or outer) problem:

$$(D) : \begin{cases} \sup_{\lambda} & d(\lambda) \\ \text{s.t.} & \lambda \succeq 0 \end{cases}$$

Consider solutions to **inner problem** as candidate solutions for P

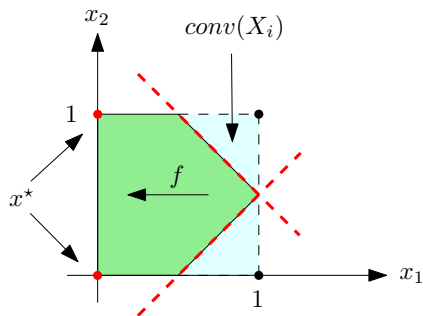
$$x_i(\lambda) \in \arg \min_{x_i \in X_i} \left\{ (c_i + \lambda' H_i) x_i \right\} \quad \lambda \in \mathbb{R}_+^m.$$

Issues affecting inner solutions



$$\begin{array}{ll} \min & -x_1 \\ \text{s.t.} & x_1 - x_2 \leq 0.5 \\ & x_1 + x_2 \leq 1.5 \\ & x_1, x_2 \in \{0, 1\} \end{array}$$

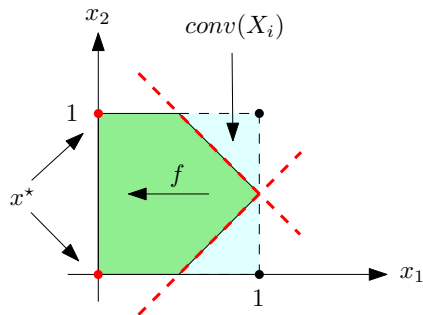
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$$\Rightarrow \max_{\lambda \geq 0} \min_{x_1, x_2 \in \{0, 1\}} \underbrace{-x_1 + \lambda_1(x_1 - x_2 - 0.5)}_{\text{red dashed line}} + \underbrace{\lambda_2(x_1 + x_2 - 1.5)}_{\text{red dashed line}}$$

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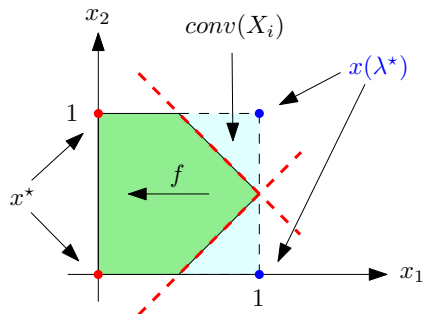


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$$\Rightarrow d^* = -1 \quad \lambda^* = (0.25, 0.25)$$

Issues affecting inner solutions

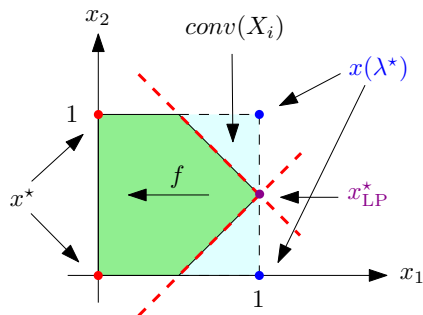


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$$\Rightarrow \max_{\lambda \geq 0} \min_{x_1, x_2 \in \{0, 1\}} -x_1 + \lambda_1(x_1 - x_2 - 0.5) + \lambda_2(x_1 + x_2 - 1.5)$$

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Main Idea

The following convexification of P is strongly related to D

$$(P_{LP}) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & x_i \in \text{conv}(X_i) \quad \forall i \in I \end{cases}$$

Known that:

- $J_{P_{LP}}^* = J_D^*$ (**surprising!**)
- $J_D^* \rightarrow J_P^*$ as $|I| \rightarrow \infty$, vanishing duality gap [Aubin '76]

We show

- 1 **an optimal solution x_{LP}^* to P_{LP} is useful**
 - ▶ feasible w.r.t. the complicating constraints
 - ▶ x_{LP}^* is non-integer in at most m subsystems
 - ▶ Shapley–Folkman–Starr Theorem
- 2 **we can compute it**, borrowing methods from conv. opt.

x_{LP}^* can be computed

- 1 solve (D) using a subgradient method

$$\begin{aligned}\lambda^{[1]} &= 0 \\ \lambda^{[k+1]} &= P_+ (\lambda^{[k]} + s^{[k]} \cdot \gamma^{[k]})\end{aligned}\tag{1}$$

- ▶ $s^{[k]}$ steplength, pick $s^{[k]} = \alpha/k$.
- ▶ $\gamma^{[k]}$ a subgradient, pick $\gamma^{[k]} = \sum_{i \in I} H_i x_i(\lambda^{[k]}) - b$.

- 2 while iterating (1), calculate average

$$\bar{x}^{[k]} = \frac{1}{k} \sum_{j=1}^k x(\lambda^{[j]})$$

Theorem (primal + dual convergence)

- $\lambda^{[k]} \rightarrow \lambda^* \in \Lambda^*$,
- if x_{LP}^* is the unique solution to \mathcal{P}_{LP} , $\bar{x}^{[k]} \rightarrow x_{LP}^*$.

Computing approximate solutions to (P)

Assume $H_i x_i \geq 0$, and $0 \in X_i$ (relaxed in [Vujanic '13]).

- choices of x_i always “consume” budget

Procedure to compute approx. \hat{x}^* to P (distributed)

- 1 compute x_{LP}^* using averaging
- 2 find $I_1 \subseteq I$ such that $(x_{LP}^*)_i \in \text{vert}(X_i)$
 - ▶ $|I_1| \geq |I| - m - 1$
- 3 for $i \in I_1$, set $\hat{x}_i^* = (x_{LP}^*)_i$
- 4 for the remaining $i \in I \setminus I_1$,
 - ▶ EITHER $\hat{x}_i^* = 0$
 - ▶ OR set $b \rightarrow b - \sum_{i \in I_1} H_i \hat{x}_i^*$, and solve smaller dimensional problem

Then, \hat{x}^* is feasible for P, and satisfies

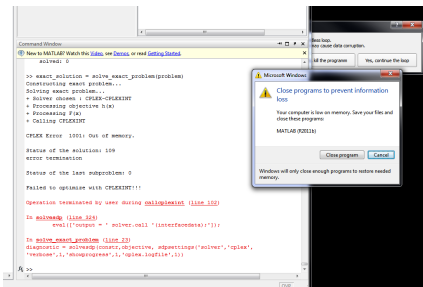
$$J_P(\hat{x}^*) - J_P^* \leq (m + 1) \max_{i \in I} \max_{x_i \in X_i} c_i^\top x_i.$$

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Partial Shipment is NP-Hard

- partial shipment problem is NP-Hard (reduction to knapsack)



```
Command Window
New to MATLAB? Watch the video, see Demos, or read Getting Started
solved: 0
>> exact_solution = solve_exact_problem(problem)
Continuing exact problem...
Solving exact problem...
+ Solving subproblem: CPLEX-CPLEXINT
+ Processing objective fix()
+ Processing fix()
+ Calling CPLEXINT
CPLEX Error 1001: Out of memory.
Status of the solution: 109
error termination
Status of the last subproblem: 0
Failed to optimize with CPLEXINT!!!
Operation terminated by user during call to cplexint (line 107)
In solve_exact (line 225)
    eval('output = ' solver.call('interfacedata');')
In solve_exact_problem (line 23)
    diagnostic = solve_exact_problem_objective_objective_settings('solver','cplex',
    'verbose',1,'showprogress',1,'cplexlogfile',1)
R>>
```

Microsoft Windows
Close programs to prevent information loss
Your computer is low on memory. Save your files and close these programs.
MATLAB (R2013b)
Close programs Cancel
Windows will only close enough programs to restore needed memory.

CPLEX with “just” $N = 300$, $M = 75$

- greedy strategies perform poorly
- tailored MINTO setup also not good (see [Dawande '06])
 - ▶ at $N = 300$, $M = 75$
avg opt. gap = 6.2% and computation time 6h

Results using Proposed Method

- generate instances of industrial size, data from [Dawande]

N	M	Proposed Method		CPLEX
		Gap (%)	Time (s)	Time (s)
100	25	0.11–0.38–0.60	16.7–18.5–21.5	10.3–63.5–202.5
300	75	0.01–0.03–0.07	61.0–62.5–63.4	–
500	50	0.04–0.06–0.11	95.0–101.0–110.6	–
600	50	0.01–0.06–0.10	113.7–116.6–119.5	–
1000	100	0.00–0.03–0.05	225.3–245.2–288.8	–

- works well because the **integer part** converges **very quickly**

Thank you for your attention!

Questions?

References:

- R. Vujanic, P. Mohajerin Esfahani, P. Goulart, S. Mariethoz and M. Morari, *Vanishing Duality Gap in Large Scale Mixed-Integer Optimization: a Solution Method with Power System Applications*, submitted to Mathematical Programming (2013).
- J. P. Aubin and I. Ekeland, *Estimates of the duality gap in nonconvex optimization*, Mathematics of Operations Research **1** (1976), no. 3, 225–245.
- Dimitri P. Bertsekas, G. Lauer, N. Sandell, and T. Posbergh, *Optimal short-term scheduling of large-scale power systems*, IEEE Transactions on Automatic Control **28** (1983), no. 1, 1– 11.
- Milind Dawande, Srinagesh Gavirneni, and Sridhar Tayur, *Effective heuristics for multiproduct partial shipment models*, Operations Research **54** (2006), no. 2, 337–352 (en).

Bkacup slides

BACKUP SLIDES

Shapley Folkman Theorem

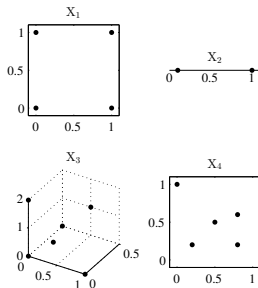
Theorem (, '69)

Let S_i , $i = 1, \dots, |I|$ be nonempty subsets of \mathbb{R}^m , with $|I| > m$, and let $S = S_1 + \dots + S_{|I|}$. Then every vector $s \in \text{conv}(S)$ can be represented as $s = s_1 + \dots + s_{|I|}$, where $s_i \in \text{conv}(S_i)$ for all $i = 1, \dots, |I|$, and $s_i \notin S_i$ for at most m indices i .

1) x_{LP}^* is useful

Visualization of the Theorem

$$\left\{ \begin{array}{l} \min_x \quad \sum_{i \in I} c_i x_i \\ \text{s.t.} \quad \sum_{i \in I} H_i x_i \preceq b \\ x_i \in X_i \quad \forall i \in I \end{array} \right.$$

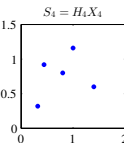
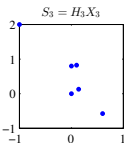
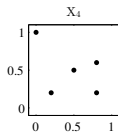
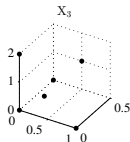
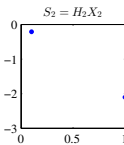
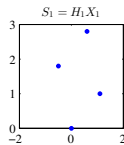
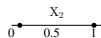
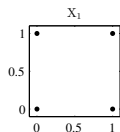


(a) subsystems X_i

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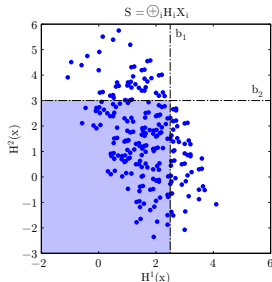
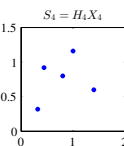
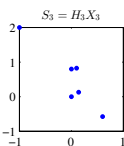
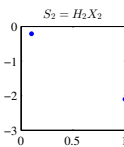
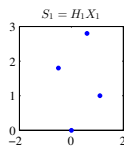
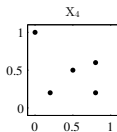
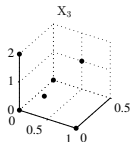
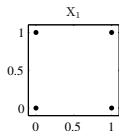
(d) subsystems X_i

(e) budget consumption
 $H_i X_i$

1) x_{LP}^* is useful

Visualization of the Theorem

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(g) subsystems X_i

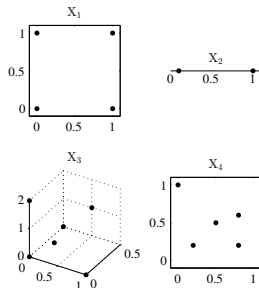
(h) budget consumption $H_i X_i$

(i) aggregated budget consumption

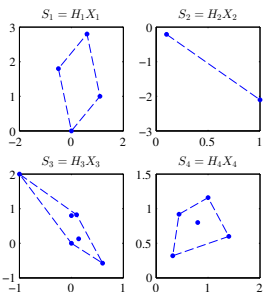
1) x_{LP}^* is useful

Visualization of the Theorem

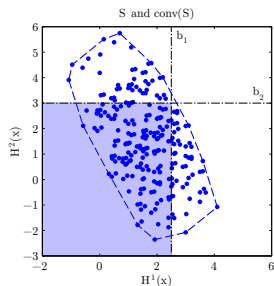
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(j) subsystems X_i



(k) budget consumption
 $H_i X_i$

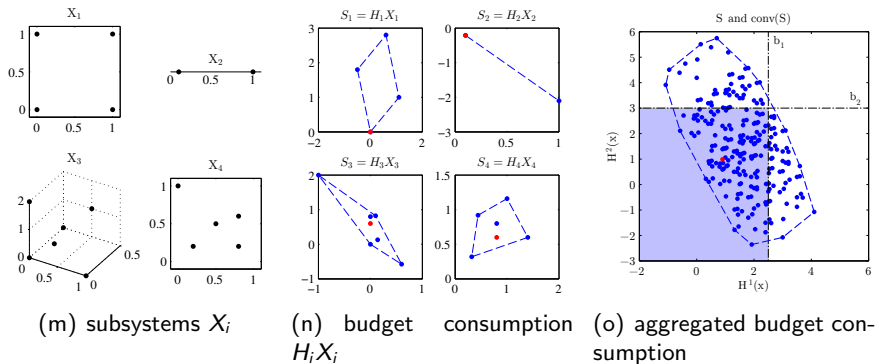


(l) aggregated budget consumption

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Visualization of the Theorem

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Result #1: Vanishing Duality Gap

- Duality for programs structured as (3) is **almost** strong:

Theorem (bound on duality gap (Ekeland '76, Bertsekas '83))

Assume that the sets X_i are non-empty and compact, and that for any given $x_i \in \text{conv}(X_i)$, there exists an $\tilde{x}_i \in X_i$ such that $H_i \tilde{x}_i \preceq H_i x_i$. Then

$$J_{\mathcal{P}}^* - J_{\mathcal{D}}^* \leq m \cdot \max_{i \in I} \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)$$

- non-convexities do not compound each other indefinitely in P , i.e. the distance between P and P_{LP} does not grow
- Thus, if $J_{\mathcal{P}}^*$ increases linearly with $|I|$,

$$\frac{J_{\mathcal{P}}^* - J_{\mathcal{D}}^*}{J_{\mathcal{P}}^*} \rightarrow 0 \quad \text{as} \quad |I| \rightarrow \infty.$$

Result #2: Primal Recovery Scheme

- immunize 3 against disturbances of m subsystems:

$$(\bar{P}) : \begin{cases} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq \bar{b} \\ & x_i \in X_i \end{cases} \quad \forall i \in I,$$

with $\bar{b} = b - \rho$ and

$$\rho^k = m \cdot \max_{i \in I} \left(\max_{x_i \in X_i} H_i^k x_i - \min_{x_i \in X_i} H_i^k x_i \right)$$

Theorem (feasible solutions)

If \bar{P}_{LP} and \bar{D} have unique solutions \bar{x}_{LP}^* and $\bar{\lambda}^*$, then any selection $x(\bar{\lambda}^*) \in \mathcal{X}(\bar{\lambda}^*)$ is feasible for 3.

- $\bar{\lambda}^* \succeq \lambda^* \rightarrow$ it is like imposing a “price uplift”
- conservatism reduction possible in some cases

Performance of the Recovered Solutions

- under some technical assumption, ...

Theorem (solutions performance)

The recovered solution $x(\bar{\lambda}^*)$ is feasible and satisfies

$$J_{\mathcal{P}}(x(\bar{\lambda}^*)) - J_{\mathcal{P}}^* \leq (m + \|\rho\|_{\infty}/\zeta) \cdot \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right)$$

- if $J_{\mathcal{P}}^*$ grows linearly with $|I|$, and X_i uniformly bounded

$$\frac{J(x(\bar{\lambda}^*)) - J_{\mathcal{P}}^*}{J_{\mathcal{P}}^*} \rightarrow 0 \quad \text{as } |I| \rightarrow \infty,$$

- in fact, optimality gap decreases at a “ $1/|I|$ ” rate

Intuitive explanation of proof that x_{LP}^* and $x_i(\lambda)$ are connected

- Problem \mathcal{P}_{LP} is equivalent to

$$\begin{aligned} \min_p \quad & \sum_{i \in I} \sum_{j \in J_i} p_{ij} (c_i x_i^j) \\ \text{s.t.} \quad & \sum_{i \in I} \sum_{j \in J_i} p_{ij} (H_i x_i^j) \leq b \\ & \sum_{j \in J_i} p_i^j = 1 \quad \forall i \in I \\ & p_i^j \geq 0 \quad \forall i \in I, j \in J_i, \end{aligned} \quad (\bar{P})$$

- Inner problems can return solution x_i^j only if its “probability” is $(p^*)_i^j > 0$.
- For $|I| - m$ subproblems only one vertex \hat{j}_i has probability > 0 (in fact, $(p^*)_i^{\hat{j}_i} = 1$), so they must be picked.
- Proof uses strict complementarity, which **requires uniqueness**.

Uniqueness assumption

“Nice” assumption

x_{LP}^* and λ^* are the unique optimal solutions to \mathcal{P}_{LP} and (D)

- sufficient but *not necessary*
- it is **stable** (for given P, it holds with probability 1)
- and **sometimes easy to check** with standard LP tools
 - ▶ e.g. problems with integrality property, such as KPs and CLPs
- adding δ_i perturbations to cost function to ensure it **substantially enhances dual convergence**
 - ▶ $\|\delta_i\|_2$ 2-3 orders of magnitude smaller than $\|c_i\|_2$ is sufficient

Detour: Strength of Relaxations (1/2)

- Assume pure binary P
- If you had to solve it, which convexification would you pick?

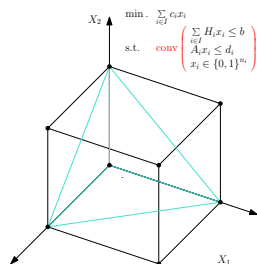
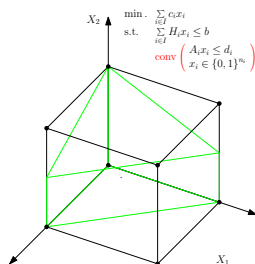
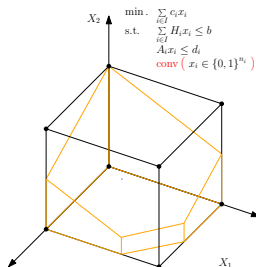
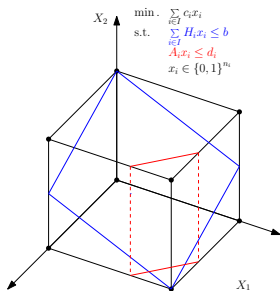
$$P_1 : \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & A_i x_i \preceq d_i \\ & \text{conv}(x_i \in \{0, 1\}^{n_i}) \end{array}$$

$$P_2 : \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \sum_{i \in I} H_i x_i \preceq b \\ & \text{conv} \left(\begin{array}{l} A_i x_i \preceq d_i \\ x_i \in \{0, 1\}^{n_i} \end{array} \right) \end{array}$$

or

$$P_3 : \begin{array}{ll} \min_x & \sum_{i \in I} c_i x_i \\ \text{s.t.} & \text{conv} \left(\begin{array}{l} \sum_{i \in I} H_i x_i \preceq b \\ A_i x_i \preceq d_i \\ x_i \in \{0, 1\}^{n_i} \end{array} \right) \end{array}$$

Detour: Strength of Relaxations (2/2)



Performance Guarantees

- Existence and uniqueness of \bar{x}_{LP}^* and $\bar{\lambda}^*$
- There exist $\zeta > 0$ and $\hat{x}_i \in \text{conv}(X_i)$ for all $i \in I$ such that

$$\sum_{i \in I} H_i \hat{x}_i \preceq b - \zeta |I| \mathbf{1}.$$

Then

$$J_{\mathcal{P}}(x(\bar{\lambda}^*)) - J_{\mathcal{P}}^* \leq (m + \|\rho\|_{\infty} / \zeta) \cdot \left(\max_{x_i \in X_i} c_i x_i - \min_{x_i \in X_i} c_i x_i \right). \quad (2)$$

This implies that, if $J_{\mathcal{P}}^*$ grows linearly with $|I|$,

$$\frac{J(x(\bar{\lambda}^*)) - J_{\mathcal{P}}^*}{J_{\mathcal{P}}^*} \rightarrow 0 \quad \text{as} \quad |I| \rightarrow \infty, \quad (3)$$

i.e. optimality gap decreases at a “ $1/|I|$ ” rate.

Dual convergence

- dual convergence ($d^{(k)} \rightarrow d^*$) is *fast*
- in all instances we tried, well within 30 dual iterations

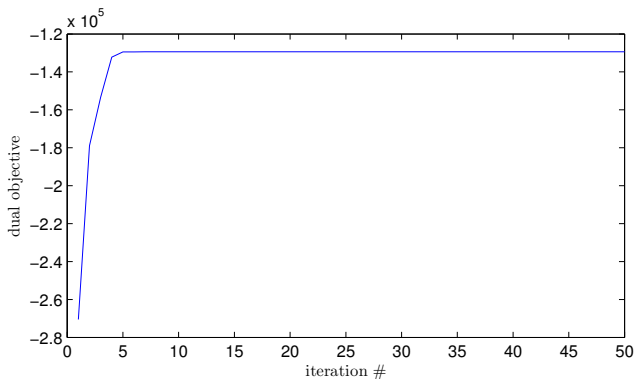


Figure: dual convergence

Ergodic Sequence: Convergence

- ergodic sequence does converge, but rather slowly
- but we don't care: we use it only to detect fractional parts

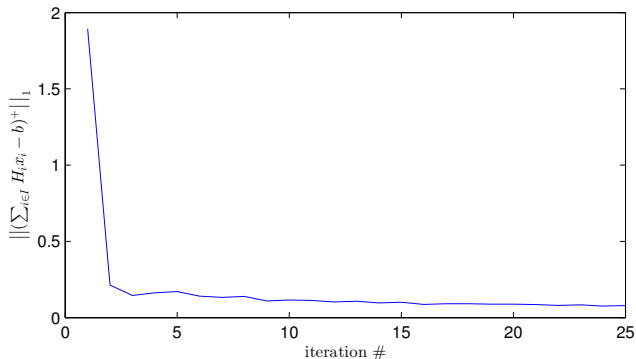


Figure: feasibility violation of ergodic sequence